



The Open
University

MST124

Essential mathematics 1

Exercise Booklet 10

1 What is a sequence?

Exercise 1

Consider the finite sequence $(a_n)_{n=1}^5$ with terms

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}.$$

- Write down the value of a_3 .
 - For which value of n is $a_n = \frac{1}{5}$?
 - Is there a value for a_6 ? Explain your answer.
 - Write down a closed form for the sequence.
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Exercise 2

For each of the sequences specified by the following closed forms and ranges of values of n , find the first three terms and the tenth term.

- $a_n = \frac{3n}{2} \quad (n = 1, 2, 3, \dots)$
 - $b_n = n - 4 \quad (n = 7, 8, 9, \dots)$
 - $c_n = \frac{n}{n+2} \quad (n = 1, 2, 3, \dots)$
 - $d_n = 5n - 3 \quad (n = 0, 1, 2, \dots)$
 - $e_n = (-3)^n \quad (n = 0, 1, 2, \dots)$
 - $f_n = -3^n \quad (n = 0, 1, 2, \dots)$
 - $g_n = (-1)^{n-1}(n+1) \quad (n = 2, 3, 4, \dots)$
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Exercise 3

- For each of the following closed forms, write down a closed form $b_n = \dots \quad (n = 0, 1, 2, \dots)$ that specifies the same sequence.

- $a_n = n \quad (n = 1, 2, 3, \dots)$
 - $a_n = \frac{n}{n+1} \quad (n = 1, 2, 3, \dots)$
 - $a_n = (-1)^n \quad (n = 1, 2, 3, \dots)$
 - $a_n = 2n - 1 \quad (n = 1, 2, 3, \dots)$
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- For each of the following closed forms, write down a closed form $b_n = \dots \quad (n = 1, 2, 3, \dots)$ that specifies the same sequence.

- $a_n = 7n + n^2 \quad (n = 0, 1, 2, \dots)$
- $a_n = 3^{n-1} \quad (n = 0, 1, 2, \dots)$
- $a_n = \frac{n}{n+2} \quad (n = 0, 1, 2, \dots)$
- $a_n = 6 - 2n \quad (n = 0, 1, 2, \dots)$

- For each of the sequences (a_n) given by the following closed forms, write down a closed form $b_n = \dots \quad (n = 0, 1, 2, \dots)$ that specifies the same sequence.

- $a_n = (n-1)(n+1) \quad (n = 2, 3, 4, \dots)$
 - $a_n = 0.1^{2n-1} \quad (n = 10, 11, 12, \dots)$
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Exercise 4

For each of the following recurrence systems, find the first five terms of the sequence specified.

- $a_1 = -5, \quad a_n = a_{n-1} + 5 \quad (n = 2, 3, 4, \dots)$
 - $b_0 = 0, \quad b_n = 2b_{n-1} + 1 \quad (n = 1, 2, 3, \dots)$
 - $c_0 = 100, \quad c_n = c_{n-1} - 50 \quad (n = 1, 2, 3, \dots)$
 - $d_1 = 1, \quad d_n = \frac{2}{d_{n-1}} \quad (n = 2, 3, 4, \dots)$
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Exercise 5

Find the first three terms specified by each of the following closed forms and ranges of values of n .

- $a_n = \frac{1}{3n} \quad (n = 3, 4, 5, \dots)$
 - $b_0 = 1, \quad b_n = \frac{1}{(b_{n-1})^2} + n \quad (n = 1, 2, 3, \dots)$
 - $c_n = \frac{n(n+1)}{2} \quad (n = 1, 2, 3, \dots)$
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Exercise 6

Explain why the specification for each of the following sequences is incomplete.

(a) $a_n = 4a_{n-1} - 3 \quad (n = 1, 2, 3, \dots)$

(b) $b_0 = 4, \quad b_n = \frac{1}{b_{n-1}}$

Exercise 7

What is wrong with the following recurrence system?

$$u_0 = 1, \quad u_n = 7u_{n-1} - 10 \quad (n = 0, 1, 2, \dots)$$

2 Arithmetic and geometric sequences

Exercise 8

Which of the following recurrence systems define arithmetic sequences? For each arithmetic sequence, write down the values of the first term a and common difference d .

(a) $x_1 = 1, \quad x_n = 2x_{n-1} \quad (n = 2, 3, 4, \dots)$

(b) $y_1 = 1, \quad y_n = y_{n-1} - 1 \quad (n = 2, 3, 4, \dots)$

(c) $z_0 = 0, \quad z_n = z_{n-1} + 1 \quad (n = 1, 2, 3, \dots)$

Exercise 9

For each of the following infinite arithmetic sequences, find the values of the first term a and common difference d , and write down the corresponding recurrence system. Calculate the fifth term in each sequence.

(a) The sequence (x_n) with first four terms

$$2, -1, -4, -7.$$

(b) The sequence (y_n) with first four terms

$$\frac{2}{3}, \frac{4}{3}, 2, \frac{8}{3}.$$

Exercise 10

For each of the following finite arithmetic sequences, find the number of terms in the sequence. Hence write down a recurrence system for the sequence.

(a) The sequence with terms

$$2, 5, 8, 11, \dots, 152.$$

(Denote the n th term by x_n .)

(b) The sequence with terms

$$6, \frac{27}{5}, \frac{24}{5}, \frac{21}{5}, \dots, -3.$$

(Denote the n th term by y_n .)

Exercise 11

Find a closed form for each of the arithmetic sequences given by the following recurrence systems. Use the closed form to calculate the 12th term in each sequence.

(a) $x_1 = 2, \quad x_n = x_{n-1} - 3 \quad (n = 2, 3, 4, \dots)$

(b) $y_1 = \frac{2}{3}, \quad y_n = y_{n-1} + \frac{2}{3} \quad (n = 2, 3, 4, \dots)$

(c) $z_1 = 4.62, \quad z_n = z_{n-1} + 0.35$
 $(n = 2, 3, 4, \dots, 20)$

(d) $u_0 = 17.5, \quad u_n = u_{n-1} - 2.5$
 $(n = 1, 2, 3, \dots)$

Exercise 12

Which of the following recurrence systems define geometric sequences? For each geometric sequence, write down the values of the first term a and common ratio r .

(a) $x_1 = 2, \quad x_n = 5(x_{n-1} + 1)$
 $(n = 2, 3, 4, \dots)$

(b) $y_0 = 10, \quad y_n = -\frac{y_{n-1}}{2} \quad (n = 1, 2, 3, \dots)$

(c) $z_1 = -6, \quad z_n = \frac{1}{z_{n-1}} \quad (n = 2, 3, 4, \dots)$

Exercise 13

For each of the following infinite geometric sequences, find the values of the first term a and common ratio r , and write down the corresponding recurrence system. Calculate the fifth term in each sequence.

- (a) The sequence (x_n) whose first four terms are
1, 3, 9, 27.
- (b) The sequence (y_n) whose first four terms are
-100, 10, -1, $\frac{1}{10}$.

Exercise 14

For each of the following finite geometric sequences, find the number of terms in the sequence. Hence write down a recurrence system for the sequence.

- (a) The sequence with terms
512, 256, 128, 64, ..., $\frac{1}{8}$.
(Denote the n th term by x_n .)
- (b) The sequence with terms
700, 770, 847,
931.7, ..., 6268.01 (to 2 d.p.).
(Denote the n th term by y_n .)

Exercise 15

Find a closed form for each of the geometric sequences given by the following recurrence systems. Use the closed form to calculate the 12th term in each sequence. In part (d) give your answer correct to three significant figures.

- (a) $x_1 = 0.001$, $x_n = 3x_{n-1}$ ($n = 2, 3, 4, \dots$)
- (b) $y_1 = -1$, $y_n = -2y_{n-1}$ ($n = 2, 3, 4, \dots$)
- (c) $z_1 = 15\,625$, $z_n = \frac{z_{n-1}}{5}$
($n = 2, 3, 4, \dots, 20$)
- (d) $u_0 = 1$, $u_n = 1.1u_{n-1}$ ($n = 1, 2, 3, \dots$)

Exercise 16

For each of the following recurrence systems, state whether the sequence specified is arithmetic, geometric or neither. If the sequence is arithmetic or geometric, find the closed form. In each case, find the seventh term in the sequence.

- (a) $x_1 = -7$, $x_n = -x_{n-1}$ ($n = 2, 3, 4, \dots$)
- (b) $y_0 = 0$, $y_1 = 1$, $y_n = y_{n-1} + y_{n-2}$
($n = 2, 3, 4, \dots$)
- (c) $z_0 = 0$, $z_n = z_{n-1} - 1$ ($n = 1, 2, 3, \dots$)

3 Graphs and long-term behaviour

Exercise 17

- (a) For each of the following closed forms and ranges of values of n , plot a graph for the first six terms of the sequence specified.
- (i) $x_n = 10 - 5n$ ($n = 1, 2, 3, \dots$)
- (ii) $y_n = 7 \times 2.1^{n-1}$ ($n = 1, 2, 3, \dots$)
- (b) For the following recurrence relation and range of values of n , plot a graph for the first six terms of the sequence specified.

$$z_1 = -16, \quad z_n = -\frac{z_{n-1}}{2} \quad (n = 2, 3, 4, \dots)$$

Exercise 18

Describe the long-term behaviour of each of the sequences given by the following closed forms.

- (a) $u_n = 10 - n$ ($n = 1, 2, 3, \dots$)
- (b) $v_n = 9 \times 3^n$ ($n = 1, 2, 3, \dots$)
- (c) $w_n = 0.1 \times 0.1^n + 18$ ($n = 1, 2, 3, \dots$)
- (d) $x_n = -3 \times 5.3^n$ ($n = 1, 2, 3, \dots$)
- (e) $y_n = -5 \left(-\frac{9}{10}\right)^n$ ($n = 1, 2, 3, \dots$)
- (f) $z_n = 2(-7.1)^n - 7.1$ ($n = 1, 2, 3, \dots$)

4 Series

Exercise 19

Find the sums of the following arithmetic series.

- (a) $1 + 4 + 7 + \cdots + 67$
 (b) $123 + 124 + 125 + \cdots + 223$
 (c) $45 + 43 + 41 + \cdots + 1$

Exercise 20

Find the sums of the following geometric series.

- (a) $5 + 5 \times 2 + 5 \times 2^2 + \cdots + 5 \times 2^7$
 (b) $32 - 16 + 8 - 4 + \cdots + \frac{1}{2}$

Exercise 21

Find the sums of the following finite series.

- (a) $1 + 2 + 3 + \cdots + 49$
 (b) $1 + 2 + 3 + \cdots + 75$
 (c) $50 + 51 + 52 + \cdots + 75$
 (d) $1^2 + 2^2 + 3^2 + \cdots + 25^2$
 (e) $15^3 + 16^3 + 17^3 + \cdots + 45^3$

Exercise 22

For each of the following infinite geometric series, determine whether or not it has a sum, and find the value of the sum if it exists.

- (a) $5 + 5\left(\frac{1}{5}\right) + 5\left(\frac{1}{5}\right)^2 + 5\left(\frac{1}{5}\right)^3 + \cdots$
 (b) $2 + 2\left(\frac{5}{4}\right) + 2\left(\frac{5}{4}\right)^2 + 2\left(\frac{5}{4}\right)^3 + \cdots$
 (c) $\frac{1}{2} - \frac{1}{2}\left(\frac{3}{4}\right) + \frac{1}{2}\left(\frac{3}{4}\right)^2 - \frac{1}{2}\left(\frac{3}{4}\right)^3 + \cdots$
 (d) $\left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4 + \left(\frac{1}{3}\right)^5 + \left(\frac{1}{3}\right)^6 + \cdots$

Exercise 23

Find the fraction equivalent to each of the following numbers.

- (a) $0.171\,717\ldots$ (b) $0.024\,624\,624\ldots$
 (c) $2.356\,435\,643\ldots$

Exercise 24

Write each of the following sums without sigma notation, giving the first three terms and the last.

- (a) $\sum_{n=0}^{99} (n+1)$ (b) $\sum_{n=2}^{15} 3^{n-1}$
 (c) $\sum_{n=0}^5 (5n+2)$

Exercise 25

Write each of the following sums in sigma notation.

- (a) $5 + 6 + 7 + \cdots + 21$
 (b) $1^5 + 2^5 + 3^5 + \cdots + 10^5$
 (c) $1 + 2^2 + 3^3 + \cdots + 9^9$

Exercise 26

Find the sums of the following finite series.

- (a) $\sum_{k=1}^{55} k$ (b) $\sum_{k=1}^{10} k^2$

Exercise 27

Find the sums of the following finite series.

- (a) $\sum_{k=1}^{15} (k+6)$ (b) $\sum_{k=1}^{22} \left(17k - \frac{k^3}{11}\right)$
 (c) $\sum_{k=7}^{12} (3+2k)$

Exercise 28

Where possible find the sums of the following series.

(a) $6 + 9 + 12 + \cdots + 78$ (b) $\sum_{k=1}^{25} (3k + 3)$

(c) $\sum_{k=1}^{\infty} \left(-\frac{3}{7}\right)^k$ (d) $\sum_{k=1}^{\infty} \left(\frac{3}{4} \times \left(-\frac{3}{4}\right)^{k-1}\right)$

(e) $\sum_{k=0}^{\infty} 0.2 \times 2^k$ (f) $\sum_{k=5}^{15} (2k^2 - k)$

(g) $\sum_{k=1}^{\infty} 3k$ (h) $\sum_{k=1}^{\infty} \left(\left(\frac{1}{2}\right)^k - \left(\frac{1}{4}\right)^k\right)$

5 The binomial theorem

Exercise 29

Use the formulas given in Section 5.1 of Unit 10 to expand the following brackets.

(a) $(X + 2Y)^2$ (b) $(1 - a^3)^2$

(c) $(x - 1)^3$ (d) $\left(\frac{x}{3y} + 3y\right)^3$

(e) $(-p - q)^4$

Exercise 30

Evaluate the following binomial coefficients without using a calculator, and then check your answer using a calculator.

(a) 8C_8 (b) 8C_0 (c) 8C_1 (d) 5C_2

(e) 9C_6 (f) ${}^{10}C_5$ (g) ${}^{101}C_{99}$

Exercise 31

Use the binomial theorem to find the first four terms in the expansion of each of the following expressions.

(a) $(x - 1)^7$ (b) $(2x + 3y)^6$

(c) $(2a - b)^9$ (d) $\left(2 - \frac{1}{2}k\right)^8$

Exercise 32

(a) Find the coefficient of x^5y^3 in the expansion of $(x - y)^8$.

(b) Find the coefficient of a^2 in the expansion of $(1 + 2a)^7$.

(c) Find the coefficient of g^6h^6 in the expansion of

$$\left(2g - \frac{1}{2}h\right)^{12}.$$

(d) Find the coefficient of y^2 in the expansion of

$$\left(y - \frac{2}{y}\right)^{16}.$$

(e) Find the coefficient of xy^2 in the expansion of

$$\left(x + \frac{y}{x}\right)^5.$$

Exercise 33

Consider the expansion of

$$\left(a^3 - \frac{2}{a^2}\right)^{10}.$$

(a) Find the coefficient of a^{25} .

(b) Find the coefficient of a^{-10} .

(c) Find the constant term.

(d) Show that there is no term in a^2 .

Solutions to exercises

Solution to Exercise 1

- (a) a_3 denotes the third term in the sequence, so $a_3 = \frac{1}{3}$.
- (b) The term $\frac{1}{5}$ is the fifth in the sequence, so if $a_n = \frac{1}{5}$, then $n = 5$.
- (c) The sequence is defined for values of n between 1 and 5, so a_6 does not exist in this case.
- (d) A closed form for the sequence is

$$a_n = \frac{1}{n} \quad (n = 1, 2, 3, 4, 5).$$

Solution to Exercise 2

- (a) $a_1 = \frac{3 \times 1}{2} = \frac{3}{2}$, $a_2 = \frac{3 \times 2}{2} = 3$,
 $a_3 = \frac{3 \times 3}{2} = \frac{9}{2}$, $a_{10} = \frac{3 \times 10}{2} = 15$.
- (b) $b_7 = 7 - 4 = 3$, $b_8 = 8 - 4 = 4$,
 $b_9 = 9 - 4 = 5$, $b_{16} = 16 - 4 = 12$.
- (c) $c_1 = \frac{1}{1+2} = \frac{1}{3}$, $c_2 = \frac{2}{2+2} = \frac{1}{2}$,
 $c_3 = \frac{3}{3+2} = \frac{3}{5}$, $c_{10} = \frac{10}{10+2} = \frac{5}{6}$.
- (d) $d_0 = 5 \times 0 - 3 = -3$, $d_1 = 5 \times 1 - 3 = 2$,
 $d_2 = 5 \times 2 - 3 = 7$, $d_9 = 5 \times 9 - 3 = 42$.
- (e) $e_0 = (-3)^0 = 1$, $e_1 = (-3)^1 = -3$,
 $e_2 = (-3)^2 = 9$, $e_9 = (-3)^9 = -19683$.
- (f) $f_0 = -3^0 = -1$, $f_1 = -3^1 = -3$,
 $f_2 = -3^2 = -9$, $f_9 = -3^9 = -19683$.
- (g) $g_2 = (-1)^{2-1}(2+1) = -3$,
 $g_3 = (-1)^{3-1}(3+1) = 4$,
 $g_4 = (-1)^{4-1}(4+1) = -5$,
 $g_{11} = (-1)^{11-1}(11+1) = 12$.

Solution to Exercise 3

- (a) (i) $b_n = n + 1 \quad (n = 0, 1, 2, \dots)$
- (ii) $b_n = \frac{n+1}{n+1+1}$
 $= \frac{n+1}{n+2} \quad (n = 0, 1, 2, \dots)$
- (iii) $b_n = (-1)^{n+1} \quad (n = 0, 1, 2, \dots)$
- (iv) $b_n = 2(n+1) - 1$
 $= 2n + 1 \quad (n = 0, 1, 2, \dots)$

- (b) (i) $b_n = 7(n-1) + (n-1)^2$
 $= n^2 + 5n - 6$
 $= (n+6)(n-1) \quad (n = 1, 2, 3, \dots)$
- (ii) $b_n = 3^{n-1-1}$
 $= 3^{n-2} \quad (n = 1, 2, 3, \dots)$
- (iii) $b_n = \frac{n-1}{n-1+2}$
 $= \frac{n-1}{n+1} \quad (n = 1, 2, 3, \dots)$
- (iv) $b_n = 6 - 2(n-1)$
 $= 6 - 2n + 2$
 $= 8 - 2n \quad (n = 1, 2, 3, \dots)$
- (c) (i) $b_n = (n+2-1)(n+2+1)$
 $= (n+1)(n+3) \quad (n = 0, 1, 2, \dots)$
- (ii) $b_n = 0.1^{2(n+10)-1}$
 $= 0.1^{(2n+19)} \quad (n = 0, 1, 2, \dots)$

Solution to Exercise 4

- (a) $a_1 = -5$, $a_2 = a_1 + 5 = 0$,
 $a_3 = a_2 + 5 = 5$, $a_4 = a_3 + 5 = 10$,
 $a_5 = a_4 + 5 = 15$.
- (b) $b_0 = 0$, $b_1 = 2 \times b_0 + 1 = 1$,
 $b_2 = 2 \times b_1 + 1 = 3$, $b_3 = 2 \times b_2 + 1 = 7$,
 $b_4 = 2 \times b_3 + 1 = 15$.
- (c) $c_0 = 100$, $c_1 = c_0 - 50 = 50$,
 $c_2 = c_1 - 50 = 0$, $c_3 = c_2 - 50 = -50$,
 $c_4 = c_3 - 50 = -100$.
- (d) $d_1 = 1$, $d_2 = 2/d_1 = 2$, $d_3 = 2/d_2 = 1$,
 $d_4 = 2/d_3 = 2$, $d_5 = 2/d_4 = 1$.

Solution to Exercise 5

- (a) $a_3 = \frac{1}{9}$, $a_4 = \frac{1}{12}$, $a_5 = \frac{1}{15}$.
- (b) $b_0 = 1$, $b_1 = \frac{1}{b_0^2} + 1 = 2$, $b_2 = \frac{1}{b_1^2} + 2 = \frac{9}{4}$.
- (c) $c_1 = \frac{1 \times (1+1)}{2} = 1$, $c_2 = \frac{2 \times (2+1)}{2} = 3$,
 $c_3 = \frac{3 \times (3+1)}{2} = 6$.

Solution to Exercise 6

- (a) When a sequence is specified using a recurrence relation such as $a_n = 4a_{n-1} - 3$, we need to know the value of the first term.

- (b) The subscript range is missing, so we don't know the range of values of n for which the recurrence system holds.

Solution to Exercise 7

The value of u_0 is specified, so the recurrence relation should define each of the subsequent terms, $u_1, u_2, u_3, \dots$, as a function of the term before. Hence, since the left-hand side of the recurrence relation is u_n , the range of values of n for the recurrence relation would normally be $n = 1, 2, 3, \dots$, not $n = 0, 1, 2, \dots$.

Solution to Exercise 8

- (a) This sequence is not arithmetic because the expression on the right of the recurrence relation is not in the form $x_{n-1} + d$. Instead, the term x_{n-1} is multiplied by 2.
- (b) This sequence is arithmetic, with parameters $a = 1$ and $d = -1$.
- (c) This sequence is arithmetic, with parameters $a = 0$ and $d = 1$.

Solution to Exercise 9

- (a) The first term is $a = 2$ and the common difference $d = -1 - 2 = -3$. So the recurrence system is
- $$x_1 = 2, \quad x_n = x_{n-1} - 3 \quad (n = 2, 3, 4, \dots).$$

The fifth term is

$$x_5 = x_4 - 3 = -7 - 3 = -10.$$

- (b) The first term is $a = \frac{2}{3}$ and the common difference $d = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$. So the recurrence system is
- $$y_1 = \frac{2}{3}, \quad y_n = y_{n-1} + \frac{2}{3} \quad (n = 2, 3, 4, \dots).$$

The fifth term is

$$y_5 = y_4 + \frac{2}{3} = \frac{8}{3} + \frac{2}{3} = \frac{10}{3}.$$

Solution to Exercise 10

- (a) For this sequence the first term is 2, the last term is 152 and the common difference d is $5 - 2 = 3$. Hence the number of terms in the sequence is

$$\frac{152 - 2}{3} + 1 = 51.$$

The corresponding recurrence system is

$$x_1 = 2, \quad x_n = x_{n-1} + 3 \\ (n = 2, 3, 4, \dots, 51).$$

- (b) For this sequence the first term is 6, the last term is -3 and the common difference d is $\frac{27}{5} - 6 = -\frac{3}{5}$. Hence the number of terms in the sequence is

$$\frac{-3 - 6}{-\frac{3}{5}} + 1 = 16.$$

The corresponding recurrence system is

$$y_1 = 6, \quad y_n = y_{n-1} - \frac{3}{5} \\ (n = 2, 3, 4, \dots, 16).$$

Solution to Exercise 11

- (a) Since $a = 2$ and $d = -3$, the closed form is

$$x_n = a + (n - 1)d \\ = 2 - 3(n - 1) \\ = 5 - 3n \quad (n = 1, 2, 3, \dots).$$

The 12th term is

$$x_{12} = 5 - 3 \times 12 = -31.$$

- (b) Since $a = \frac{2}{3}$ and $d = \frac{2}{3}$, the closed form is

$$y_n = \frac{2}{3} + \frac{2}{3}(n - 1) \\ = \frac{2}{3}n \quad (n = 1, 2, 3, \dots).$$

The 12th term is

$$y_{12} = \frac{2}{3} \times 12 = 8.$$

- (c) Since $a = 4.62$ and $d = 0.35$, the closed form is

$$z_n = 4.62 + 0.35(n - 1) \\ = 0.35n + 4.27 \quad (n = 1, 2, 3, \dots, 20).$$

The 12th term is

$$z_{12} = 0.35 \times 12 + 4.27 = 8.47.$$

- (d) In this case, the first term is u_0 not u_1 so the alternative closed form is used. Since $a = 17.5$ and $d = -2.5$, the closed form is

$$u_n = a + nd \\ = 17.5 - 2.5n \quad (n = 0, 1, 2, \dots).$$

The 12th term is

$$u_{11} = 17.5 - 2.5 \times 11 = -10.$$

Solution to Exercise 12

- (a) The sequence (x_n) is not geometric because the expression on the right of the recurrence relation is not of the form rx_{n-1} .
- (b) The sequence (y_n) is geometric, with parameters $a = 10$ and $r = -\frac{1}{2}$.
- (c) The sequence (z_n) is not geometric because the expression on the right of the recurrence relation is not of the form rz_{n-1} .

Solution to Exercise 13

- (a) The first term is $a = 1$ and the common ratio is $r = 3/1 = 3$. So the recurrence system is

$$x_1 = 1, \quad x_n = 3x_{n-1} \quad (n = 2, 3, 4, \dots).$$

The fifth term is

$$x_5 = 3 \times x_4 = 3 \times 27 = 81.$$

- (b) The first term is $a = -100$ and the common ratio is $r = 10/(-100) = -\frac{1}{10}$. So the recurrence system is

$$y_1 = -100, \quad y_n = -\frac{y_{n-1}}{10} \quad (n = 2, 3, 4, \dots).$$

The fifth term is

$$y_5 = -\frac{y_4}{10} = -\frac{1}{100}.$$

Solution to Exercise 14

- (a) Suppose that the sequence has N terms, with first term $x_1 = 512$. Then the last term is $x_N = \frac{1}{8}$. The common ratio is

$$r = \frac{x_2}{x_1} = \frac{256}{512} = \frac{1}{2}.$$

Now the last term is obtained from

$$x_N = x_1 r^{N-1},$$

so

$$r^{N-1} = \frac{x_N}{x_1},$$

that is,

$$\left(\frac{1}{2}\right)^{N-1} = \frac{\frac{1}{8}}{512} = \frac{1}{4096}.$$

Taking the natural logarithm of both sides of this equation gives

$$(N-1) \ln\left(\frac{1}{2}\right) = \ln\left(\frac{1}{4096}\right),$$

from which

$$N = 1 + \frac{\ln(1/4096)}{\ln(1/2)} = 13.$$

Hence there are 13 terms in the sequence. The corresponding recurrence system is

$$x_1 = 512, \quad x_n = \frac{x_{n-1}}{2} \quad (n = 2, 3, 4, \dots, 13).$$

- (b) Suppose that the sequence has N terms, with first term $y_1 = 700$. Then the last term (to 2 d.p.) is $y_N = 6268.01$. The common ratio is

$$r = \frac{y_2}{y_1} = \frac{770}{700} = 1.1.$$

Now the last term is obtained from

$$y_N = y_1 r^{N-1},$$

so

$$r^{N-1} = \frac{y_N}{y_1},$$

that is,

$$1.1^{N-1} = \frac{y_N}{700},$$

where $y_N = 6268.01$ (to 2 d.p.). Taking the natural logarithm of both sides of this equation gives

$$(N-1) \ln 1.1 = \ln(y_N/700),$$

from which

$$N = 1 + \frac{\ln(y_N/700)}{\ln 1.1}.$$

Substituting the approximate value $y_N = 6268.01$ into this equation gives

$$N = 23.999\,997\dots$$

(This non-integer outcome for N arises from the fact that the value of y_N is not exact.)

Hence there are 24 terms in the sequence. The corresponding recurrence system is

$$y_1 = 700, \quad y_n = 1.1y_{n-1} \quad (n = 2, 3, 4, \dots, 24).$$

Solution to Exercise 15

- (a) Since $a = 0.001$ and $r = 3$, the closed form is

$$\begin{aligned}x_n &= ar^{n-1} \\&= 0.001 \times 3^{n-1} \quad (n = 1, 2, 3, \dots).\end{aligned}$$

The 12th term is

$$x_{12} = 0.001 \times 3^{11} = 177.147.$$

- (b) Since $a = -1$ and $r = -2$, the closed form is

$$y_n = -(-2)^{n-1} \quad (n = 1, 2, 3, \dots).$$

The 12th term is

$$y_{12} = -(-2)^{11} = 2048.$$

- (c) Since $a = 15\,625$ and $r = \frac{1}{5}$, the closed form is

$$\begin{aligned}z_n &= 15\,625 \left(\frac{1}{5}\right)^{n-1} \\&\quad (n = 1, 2, 3, \dots, 20).\end{aligned}$$

The 12th term is

$$z_{12} = 15\,625 \left(\frac{1}{5}\right)^{11} = \frac{1}{3125}.$$

- (d) In this case the first term is u_0 not u_1 , so the alternative closed form is used. Since $a = 1$ and $r = 1.1$, the closed form is

$$\begin{aligned}u_n &= ar^n \\&= 1 \times 1.1^n = 1.1^n \quad (n = 0, 1, 2, \dots).\end{aligned}$$

The 12th term is

$$x_{11} = 1.1^{11} = 2.85 \text{ (to 3 s.f.)}.$$

Solution to Exercise 16

- (a) This is a geometric sequence with first term $a = -7$ and common ratio $r = -1$. The closed form is

$$\begin{aligned}x_n &= -7(-1)^{n-1} \\&= 7(-1)^n \quad (n = 1, 2, 3, \dots).\end{aligned}$$

The seventh term is

$$x_7 = 7(-1)^7 = -7.$$

- (b) This is neither an arithmetic nor a geometric sequence. Each new term is formed by the sum of the previous two terms.

The sequence continues

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \dots,$$

so the seventh term is 8. This sequence is known as the Fibonacci sequence.

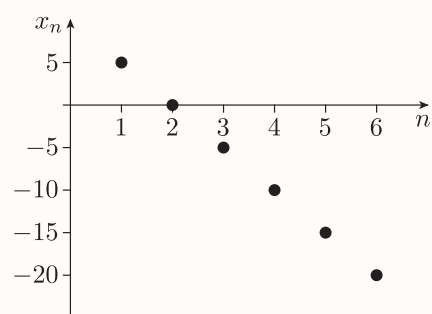
- (c) This is an arithmetic sequence with first term $a = 0$ and common difference $d = -1$. The first term is z_0 rather than z_1 , so the closed form is

$$\begin{aligned}z_n &= 0 + (-1) \times n \\&= -n \quad (n = 0, 1, 2, \dots).\end{aligned}$$

The seventh term is $z_6 = -6$.

Solution to Exercise 17

- (a) (i) The first point to be plotted is $(1, x_1) = (1, 5)$. The second point is $(2, x_2) = (2, 0)$. The subsequent points are $(3, -5)$, $(4, -10)$, $(5, -15)$ and $(6, -20)$. The graph is as follows.



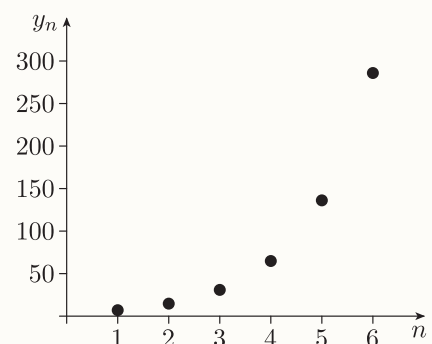
- (ii) The first point to be plotted is

$$(1, y_1) = (1, 7 \times 2.1^0) = (1, 7).$$

The second point is

$$(2, y_2) = (2, 7 \times 2.1^1) = (2, 14.7).$$

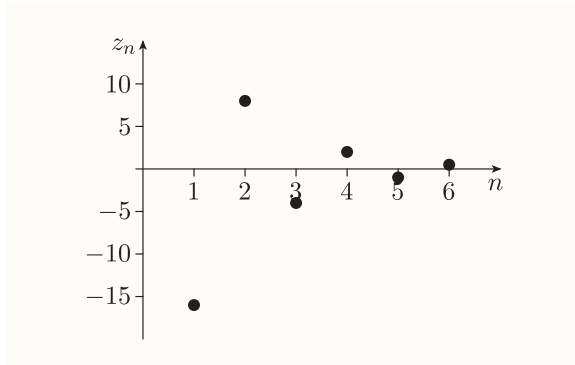
The subsequent points (to 1 d.p.) are $(3, 30.9)$, $(4, 64.8)$, $(5, 136.1)$ and $(6, 285.9)$. The graph is as follows.



- (b) The first point to be plotted is $(1, z_1) = (1, -16)$. The second point is

$$(2, z_2) = \left(2, -\frac{z_1}{2}\right) = (2, 8).$$

The subsequent points are $(3, -4)$, $(4, 2)$, $(5, -1)$ and $(6, \frac{1}{2})$. The graph is as follows.



Solution to Exercise 18

- (a) The sequence (u_n) is arithmetic with common difference $d = -1$. Since $d < 0$, the sequence is decreasing and $u_n \rightarrow -\infty$ as $n \rightarrow \infty$.
- (b) Since $3 > 1$, the sequence (3^n) is increasing and $3^n \rightarrow \infty$ as $n \rightarrow \infty$. To obtain (v_n) we multiply each term by the positive constant 9. Hence (v_n) is increasing and $v_n \rightarrow \infty$ as $n \rightarrow \infty$.
- (c) Since $0 < 0.1 < 1$, the sequence (0.1^n) is decreasing and $0.1^n \rightarrow 0$ as $n \rightarrow \infty$. Hence the sequence (0.1×0.1^n) is also decreasing with limit 0. To obtain (w_n) we add 18 to each term in this sequence, so (w_n) is decreasing and $w_n \rightarrow 18$ as $n \rightarrow \infty$.
- (d) Since $5.3 > 1$, the sequence (5.3^n) is increasing and $5.3^n \rightarrow \infty$ as $n \rightarrow \infty$. To obtain (x_n) we multiply each term by the negative constant -3 . Hence (x_n) is decreasing and $x_n \rightarrow -\infty$ as $n \rightarrow \infty$.
- (e) Since $-1 < -\frac{9}{10} < 0$, the sequence $((-\frac{9}{10})^n)$ alternates in sign and $(-\frac{9}{10})^n \rightarrow 0$ as $n \rightarrow \infty$. To obtain (y_n) we multiply each term by the negative constant -5 . Hence (y_n) also alternates in sign and $y_n \rightarrow 0$ as $n \rightarrow \infty$.
- (f) Since $-7.1 < -1$, the sequence $((-7.1)^n)$ alternates in sign and is unbounded. Multiplying each term by the positive constant 2 and subtracting 7.1 does not affect this behaviour, so (z_n) also alternates in sign and is unbounded.

Solution to Exercise 19

In each case we use the expression

$$\frac{1}{2}n(2a + (n-1)d)$$

for the sum of the series, where a is the first term, d is the common difference and n is the number of terms with

$$n = \frac{\text{last term} - \text{first term}}{\text{common difference}} + 1.$$

- (a) Here $a = 1$, $d = 3$ and

$$n = \frac{67 - 1}{3} + 1 = 23.$$

Hence the sum is

$$\begin{aligned} \frac{1}{2} \times 23 \times (2 \times 1 + (23-1) \times 3) \\ = \frac{1}{2} \times 23 \times 68 \\ = 782. \end{aligned}$$

- (b) Here $a = 123$, $d = 1$ and

$$n = \frac{223 - 123}{1} + 1 = 101.$$

Hence the sum is

$$\begin{aligned} \frac{1}{2} \times 101 \times (2 \times 123 + (101-1) \times 1) \\ = \frac{1}{2} \times 101 \times 346 \\ = 17\,473. \end{aligned}$$

- (c) Here $a = 45$, $d = -2$ and

$$n = \frac{1 - 45}{(-2)} + 1 = 23.$$

Hence the sum is

$$\begin{aligned} \frac{1}{2} \times 23 \times (2 \times 45 + (23-1) \times (-2)) \\ = \frac{1}{2} \times 23 \times 46 \\ = 529. \end{aligned}$$

Solution to Exercise 20

- (a) Each term in the series has the form $5 \times 2^{n-1}$. Therefore, the first term is $a = 5$, the common ratio is $r = 2$ and the number of terms is $n = 8$. Hence the sum is

$$\frac{a(1 - r^n)}{1 - r} = \frac{5(1 - 2^8)}{1 - 2} = 1275.$$

- (b) The terms of the series form a geometric sequence with closed form

$$x_n = 32 \left(-\frac{1}{2}\right)^{n-1} \quad (n = 1, 2, 3, \dots, 7).$$

(The last term is obtained by multiplying 32 by $-\frac{1}{2}$ six times, so the number of terms is $n = 7$). The first term is $a = 32$ and the common ratio is $r = -\frac{1}{2}$. Hence the sum is

$$\begin{aligned} \frac{a(1-r^n)}{1-r} &= \frac{32 \left(1 - \left(-\frac{1}{2}\right)^7\right)}{1 - \left(-\frac{1}{2}\right)} \\ &= \frac{129/4}{3/2} \\ &= \frac{43}{2}. \end{aligned}$$

Solution to Exercise 21

- (a) The formula for the first n natural numbers is

$$1 + 2 + 3 + \dots + n = \frac{1}{2}n(n+1).$$

Here $n = 49$, and so

$$1 + 2 + 3 + \dots + 49 = \frac{1}{2} \times 49 \times 50 = 1225.$$

- (b) We use the same formula as in part (a). Here $n = 75$, and so

$$1 + 2 + 3 + \dots + 75 = \frac{1}{2} \times 75 \times 76 = 2850.$$

- (c) Using the results of parts (a) and (b) gives

$$\begin{aligned} &50 + 51 + 52 + \dots + 75 \\ &= (1 + 2 + 3 + \dots + 75) \\ &\quad - (1 + 2 + 3 + \dots + 49) \\ &= 2850 - 1225 \\ &= 1625. \end{aligned}$$

- (d) The formula for the squares of the first n natural numbers is

$$\begin{aligned} &1^2 + 2^2 + 3^2 + \dots + n^2 \\ &= \frac{1}{6}n(n+1)(2n+1). \end{aligned}$$

Here $n = 25$, and so

$$\begin{aligned} &1^2 + 2^2 + 3^2 + \dots + 25^2 \\ &= \frac{1}{6} \times 25 \times 26 \times 51 \\ &= 5525. \end{aligned}$$

- (e) The sum can be written as

$$\begin{aligned} &15^3 + 16^3 + 17^3 + \dots + 45^3 \\ &= (1^3 + 2^3 + 3^3 + \dots + 45^3) \\ &\quad - (1^3 + 2^3 + 3^3 + \dots + 14^3). \end{aligned}$$

The formula for the cubes of the first n natural numbers is

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{1}{4}n^2(n+1)^2.$$

So

$$\begin{aligned} &15^3 + 16^3 + 17^3 + \dots + 45^3 \\ &= \frac{1}{4} \times 45^2 \times 46^2 - \frac{1}{4} \times 14^2 \times 15^2 \\ &= 1\,071\,225 - 11\,025 \\ &= 1\,060\,200. \end{aligned}$$

Solution to Exercise 22

- (a) This is an infinite geometric series with first term $a = 5$ and common ratio $r = \frac{1}{5}$. Since $-1 < r < 1$, the series has a sum, namely

$$\frac{a}{1-r} = \frac{5}{1-\frac{1}{5}} = \frac{25}{4}.$$

- (b) This is an infinite geometric series with first term $a = 2$ and common ratio $r = \frac{5}{4}$. Since $r > 1$, the series does not have a sum.

- (c) This is an infinite geometric series with first term $a = \frac{1}{2}$ and common ratio $r = -\frac{3}{4}$. Since $-1 < r < 1$, the series has a sum, namely

$$\frac{a}{1-r} = \frac{\frac{1}{2}}{1-\left(-\frac{3}{4}\right)} = \frac{\frac{1}{2}}{\frac{7}{4}} = \frac{2}{7}.$$

- (d) This is an infinite geometric series with first term $a = \left(\frac{1}{3}\right)^3$ and common ratio $r = \frac{1}{3}$. Since $-1 < r < 1$, the series has a sum, namely

$$\frac{a}{1-r} = \frac{\left(\frac{1}{3}\right)^3}{1-\frac{1}{3}} = \frac{\frac{1}{27}}{\frac{2}{3}} = \frac{1}{18}.$$

Solution to Exercise 23

- (a) Let $s = 0.171\,717\ldots$. The repeating group, '17', is 2 digits long, so multiply s by 10^2 , to obtain

$$100s = 17.171\,717\ldots = 17 + s.$$

Hence $99s = 17$ and $s = \frac{17}{99}$.

So

$$0.171\,717\ldots = \frac{17}{99}.$$

- (b) Let $s = 0.024\,624\,624\ldots$. To deal with the zero after the decimal point, multiply s by 10 to give

$$10s = 0.246\,246\,246\ldots$$

The repeating group, '246', is 3 digits long, so multiply each side by 10^3 , to obtain

$$\begin{aligned} 10\,000s &= 246.246\,246\,246\ldots \\ &= 246 + 10s. \end{aligned}$$

Hence $9990s = 246$ and $s = \frac{246}{9990} = \frac{41}{1665}$.
So

$$0.024\,624\,624\ldots = \frac{41}{1665}.$$

- (c) Let $s = 0.356\,435\,643\ldots$. The repeating group, '3564', is 4 digits long, so multiply s by 10^4 , to obtain

$$\begin{aligned} 10\,000s &= 3564.356\,435\,643\ldots \\ &= 3564 + s. \end{aligned}$$

Hence $9999s = 3564$ and $s = \frac{3564}{9999} = \frac{36}{101}$.
So

$$2.356\,435\,643\ldots = 2 + \frac{36}{101} = \frac{238}{101}.$$

Solution to Exercise 24

- (a) $\sum_{n=0}^{99} (n+1) = 1 + 2 + 3 + \cdots + 100$
- (b) $\sum_{n=2}^{15} 3^{n-1} = 3 + 3^2 + 3^3 + \cdots + 3^{14}$
- (c) $\sum_{n=0}^5 (5n+2) = 2 + 7 + 12 + \cdots + 27$

Solution to Exercise 25

- (a) $5 + 6 + 7 + \cdots + 21 = \sum_{n=5}^{21} n$
- (b) $1^5 + 2^5 + 3^5 + \cdots + 10^5 = \sum_{n=1}^{10} n^5$
- (c) $1 + 2^2 + 3^3 + \cdots + 9^9 = \sum_{n=1}^9 n^n$

Solution to Exercise 26

- (a) Using $\sum_{k=1}^n k = \frac{1}{2}n(n+1)$, we have

$$\sum_{k=1}^{55} k = \frac{1}{2} \times 55 \times (55+1) = 1540.$$

- (b) Using $\sum_{k=1}^n k^2 = \frac{1}{6}n(n+1)(2n+1)$, we

have

$$\begin{aligned} \sum_{k=1}^{10} k^2 &= \frac{1}{6} \times 10 \times (10+1) \times (2 \times 10 + 1) \\ &= \frac{1}{6} \times 10 \times 11 \times 21 \\ &= 385. \end{aligned}$$

Solution to Exercise 27

- (a) Using $\sum_{k=1}^n k = \frac{1}{2}n(n+1)$ and $\sum_{k=1}^n 1 = n$, we have

$$\begin{aligned} \sum_{k=1}^{15} (k+6) &= \sum_{k=1}^{15} k + 6 \sum_{k=1}^{15} 1 \\ &= \frac{1}{2} \times 15 \times 16 + 6 \times 15 \\ &= 120 + 90 \\ &= 210. \end{aligned}$$

- (b) Using $\sum_{k=1}^n k = \frac{1}{2}n(n+1)$ and

$$\sum_{k=1}^n k^3 = \frac{1}{4}n^2(n+1)^2, \text{ we have}$$

$$\begin{aligned} \sum_{k=1}^{22} \left(17k - \frac{k^3}{11} \right) &= 17 \sum_{k=1}^{22} k - \frac{1}{11} \sum_{k=1}^{22} k^3 \\ &= 17 \times \frac{1}{2} \times 22 \times 23 \\ &\quad - \frac{1}{11} \times \frac{1}{4} \times 22^2 \times 23^2 \\ &= 4301 - 5819 \\ &= -1518. \end{aligned}$$

(c) We have

$$\sum_{k=7}^{12} (3 + 2k) = \sum_{k=1}^{12} (3 + 2k) - \sum_{k=1}^6 (3 + 2k).$$

Now

$$\begin{aligned} \sum_{k=1}^{12} (3 + 2k) &= 3 \sum_{k=1}^{12} 1 + 2 \sum_{k=1}^{12} k \\ &= 3 \times 12 + 2 \times \frac{1}{2} \times 12 \times 13 \\ &= 36 + 156 \\ &= 192. \end{aligned}$$

Similarly,

$$\begin{aligned} \sum_{k=1}^6 (3 + 2k) &= 3 \sum_{k=1}^6 1 + 2 \sum_{k=1}^6 k \\ &= 3 \times 6 + 2 \times \frac{1}{2} \times 6 \times 7 \\ &= 18 + 42 \\ &= 60. \end{aligned}$$

Hence

$$\sum_{k=7}^{12} (3 + 2k) = 192 - 60 = 132.$$

Solution to Exercise 28

(a) This is a finite arithmetic series with first term $a = 6$, common difference $d = 3$ and number of terms

$$n = \frac{78 - 6}{3} + 1 = 25.$$

So the sum of the series is

$$\begin{aligned} &\frac{1}{2}n(2a + (n - 1)d) \\ &= \frac{1}{2} \times 25 \times (2 \times 6 + (25 - 1) \times 3) \\ &= 1050. \end{aligned}$$

(b) This is the same series as in part (a), with first term $a = 3 \times 1 + 3 = 6$, common difference $d = 3$ and number of terms $n = 25$. So the sum of the series is 1050.

(c) This is an infinite geometric series with first term $a = \left(-\frac{3}{7}\right)^1 = -\frac{3}{7}$ and common ratio $r = -\frac{3}{7}$.

Since $-1 < r < 1$, the series has sum

$$\frac{a}{1 - r} = \frac{-\frac{3}{7}}{1 - \left(-\frac{3}{7}\right)} = \frac{-\frac{3}{7}}{\frac{10}{7}} = -\frac{3}{10}.$$

(d) This is an infinite geometric series with first term $a = \frac{3}{4} \times \left(-\frac{3}{4}\right)^{1-1} = \frac{3}{4}$ and common ratio $r = -\frac{3}{4}$. Since $-1 < r < 1$, the series has sum

$$\frac{a}{1 - r} = \frac{\frac{3}{4}}{1 - \left(-\frac{3}{4}\right)} = \frac{\frac{3}{4}}{\frac{7}{4}} = \frac{3}{7}.$$

(e) This is an infinite geometric series with first term $a = 0.2$ and common ratio $r = 2$. Since $r > 1$, the series doesn't have a sum.

(f) This is a finite series composed of the standard series

$$\sum_{k=1}^n k^2 = \frac{1}{6}n(n+1)(2n+1) \text{ and } \sum_{k=1}^n k = \frac{1}{2}n(n+1). \text{ We have}$$

$$\begin{aligned} \sum_{k=5}^{15} (2k^2 - k) &= \sum_{k=1}^{15} (2k^2 - k) \\ &\quad - \sum_{k=1}^4 (2k^2 - k). \end{aligned}$$

Now

$$\begin{aligned} \sum_{k=1}^{15} (2k^2 - k) &= 2 \sum_{k=1}^{15} k^2 - \sum_{k=1}^{15} k \\ &= 2 \times \frac{1}{6} \times 15 \times 16 \times 31 \\ &\quad - \frac{1}{2} \times 15 \times 16 \\ &= 2480 - 120 \\ &= 2360. \end{aligned}$$

Similarly,

$$\begin{aligned} \sum_{k=1}^4 (2k^2 - k) &= 2 \sum_{k=1}^4 k^2 - \sum_{k=1}^4 k \\ &= 2 \times \frac{1}{6} \times 4 \times 5 \times 9 \\ &\quad - \frac{1}{2} \times 4 \times 5 \\ &= 60 - 10 \\ &= 50. \end{aligned}$$

Hence

$$\sum_{k=5}^{15} (2k^2 - k) = 2360 - 50 = 2310.$$

- (g) This is an infinite arithmetic series with first term $a = 3 \times 1 = 3$ and common difference $d = 3$. An infinite arithmetic series has a sum only when the first term and common difference are both zero, so this series doesn't have a sum.

- (h) Using the rules for manipulating series gives

$$\sum_{k=1}^{\infty} \left(\left(\frac{1}{2} \right)^k - \left(\frac{1}{4} \right)^k \right) = \sum_{k=1}^{\infty} \left(\frac{1}{2} \right)^k - \sum_{k=1}^{\infty} \left(\frac{1}{4} \right)^k.$$

These are infinite geometric series with $a = r = \frac{1}{2}$ and $a = r = \frac{1}{4}$, respectively. In both cases, $-1 < r < 1$ so the sum is given by $a/(1-r)$. Hence

$$\begin{aligned} \sum_{k=1}^{\infty} \left(\left(\frac{1}{2} \right)^k - \left(\frac{1}{4} \right)^k \right) &= \frac{\frac{1}{2}}{1 - \frac{1}{2}} - \frac{\frac{1}{4}}{1 - \frac{1}{4}} \\ &= 1 - \frac{1}{3} \\ &= \frac{2}{3}. \end{aligned}$$

Solution to Exercise 29

- (a) Using the formula

$$(a+b)^2 = a^2 + 2ab + b^2$$

and substituting $a = X$ and $b = 2Y$ gives

$$\begin{aligned} (X + 2Y)^2 &= X^2 + 2X \times 2Y + (2Y)^2 \\ &= X^2 + 4XY + 4Y^2. \end{aligned}$$

- (b) Using the formula

$$(a+b)^2 = a^2 + 2ab + b^2$$

and substituting $a = 1$ and $b = -a^3$ gives

$$\begin{aligned} (1 - a^3)^2 &= 1^2 + 2 \times 1 \times (-a^3) + (-a^3)^2 \\ &= 1 - 2a^3 + a^6. \end{aligned}$$

- (c) Using the formula

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

and substituting $a = x$ and $b = -1$ gives

$$\begin{aligned} (x-1)^3 &= x^3 + 3x^2(-1) + 3x(-1)^2 + (-1)^3 \\ &= x^3 - 3x^2 + 3x - 1. \end{aligned}$$

- (d) Using the formula

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

and substituting $a = x/3y$ and $b = 3y$ gives

$$\begin{aligned} \left(\frac{x}{3y} + 3y \right)^3 &= \left(\frac{x}{3y} \right)^3 + 3 \left(\frac{x}{3y} \right)^2 (3y) \\ &\quad + 3 \left(\frac{x}{3y} \right) (3y)^2 + (3y)^3 \\ &= \frac{x^3}{27y^3} + \frac{x^2}{y} + 9xy + 27y^3. \end{aligned}$$

- (e) Using the formula

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

and substituting $a = -p$ and $b = -q$ gives

$$\begin{aligned} (-p-q)^4 &= (-p)^4 + 4(-p)^3(-q) + 6(-p)^2(-q)^2 \\ &\quad + 4(-p)(-q)^3 + (-q)^4 \\ &= p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4. \end{aligned}$$

Solution to Exercise 30

- (a) Using

$${}^nC_k = \frac{n!}{k!(n-k)!},$$

we have

$${}^8C_8 = \frac{8!}{8!(8-8)!} = \frac{8!}{8!} = 1.$$

$$(b) {}^8C_0 = \frac{8!}{0!(8-0)!} = \frac{8!}{8!} = 1$$

$$(c) {}^8C_1 = \frac{8}{1} = 8$$

$$(d) {}^5C_2 = \frac{5 \times 4}{2 \times 1} = 10$$

$$(e) {}^9C_6 = {}^9C_3 = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84$$

$$(f) {}^{10}C_5 = \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} = 252$$

$$(g) {}^{101}C_{99} = {}^{101}C_2 = \frac{101 \times 100}{2 \times 1} = 5050$$

Solution to Exercise 31

- (a) By the binomial theorem with $n = 7$, $a = x$ and $b = -1$, the first four terms in the expansion of $(x - 1)^7$ are

$$\begin{aligned} & x^7 + {}^7C_1x^6(-1) + {}^7C_2x^5(-1)^2 \\ & \quad + {}^7C_3x^4(-1)^3 \\ & = x^7 - 7x^6 + \frac{7 \times 6}{2!}x^5 - \frac{7 \times 6 \times 5}{3!}x^4 \\ & = x^7 - 7x^6 + 21x^5 - 35x^4. \end{aligned}$$

- (b) By the binomial theorem with $n = 6$, $a = 2x$ and $b = 3y$, the first four terms in the expansion of $(2x + 3y)^6$ are

$$\begin{aligned} & (2x)^6 + {}^6C_1(2x)^5(3y) + {}^6C_2(2x)^4(3y)^2 \\ & \quad + {}^6C_3(2x)^3(3y)^3 \\ & = 64x^6 + 6 \times 2^5 \times 3 \times x^5y \\ & \quad + \frac{6 \times 5}{2!} \times 2^4 \times 3^2 \times x^4y^2 \\ & \quad + \frac{6 \times 5 \times 4}{3!} \times 2^3 \times 3^3 \times x^3y^3 \\ & = 64x^6 + 576x^5y + 2160x^4y^2 + 4320x^3y^3. \end{aligned}$$

- (c) By the binomial theorem with $n = 9$, and with a replaced by $2a$ and b replaced by $-b$, the first four terms in the expansion of $(2a - b)^9$ are

$$\begin{aligned} & (2a)^9 + {}^9C_1(2a)^8(-b) + {}^9C_2(2a)^7(-b)^2 \\ & \quad + {}^9C_3(2a)^6(-b)^3 \\ & = 512a^9 - 9 \times 2^8 \times a^8b + \frac{9 \times 8}{2!} \times 2^7 \times a^7b^2 \\ & \quad - \frac{9 \times 8 \times 7}{3!} \times 2^6 \times a^6b^3 \\ & = 512a^9 - 2304a^8b + 4608a^7b^2 - 5376a^6b^3. \end{aligned}$$

- (d) By the binomial theorem with $n = 8$, $a = 2$ and $b = -\frac{1}{2}k$, the first four terms in the expansion of $(2 - \frac{1}{2}k)^8$ are

$$\begin{aligned} & 2^8 + {}^8C_1 \times 2^7 \left(-\frac{1}{2}k\right) + {}^8C_2 \times 2^6 \left(-\frac{1}{2}k\right)^2 \\ & \quad + {}^8C_3 \times 2^5 \left(-\frac{1}{2}k\right)^3 \\ & = 256 - 8 \times 2^7 \left(\frac{1}{2}\right)k + \frac{8 \times 7}{2!} \times 2^6 \left(\frac{1}{2}\right)^2 k^2 \\ & \quad - \frac{8 \times 7 \times 6}{3!} \times 2^5 \left(\frac{1}{2}\right)^3 k^3 \\ & = 256 - 512k + 448k^2 - 224k^3. \end{aligned}$$

Solution to Exercise 32

- (a) By the binomial theorem with $n = 8$, $a = x$ and $b = -y$, each term in the expansion is of the form

$${}^8C_k x^{8-k} (-y)^k = {}^8C_k (-1)^k x^{8-k} y^k.$$

The term in x^5y^3 is obtained when $k = 3$. Hence the coefficient of x^5y^3 is

$${}^8C_3 (-1)^3 = 56 \times (-1) = -56.$$

- (b) By the binomial theorem with $n = 7$, $a = 1$ and $b = 2a$, each term in the expansion is of the form

$${}^7C_k (1)^{7-k} (2a)^k = {}^7C_k \times 2^k a^k.$$

The term in a^2 is obtained when $k = 2$. Hence the coefficient of a^2 is

$${}^7C_2 \times 2^2 = 21 \times 4 = 84.$$

- (c) By the binomial theorem with $n = 12$, $a = 2g$ and $b = -\frac{1}{2}h$, each term in the expansion is of the form

$$\begin{aligned} & {}^{12}C_k (2g)^{12-k} \left(-\frac{1}{2}h\right)^k \\ & = {}^{12}C_k \times 2^{12-k} \left(-\frac{1}{2}\right)^k g^{12-k} h^k. \end{aligned}$$

The term in g^6h^6 is obtained when $k = 6$. Hence the coefficient of g^6h^6 is

$${}^{12}C_6 \times 2^6 \left(-\frac{1}{2}\right)^6 = {}^{12}C_6 = 924.$$

- (d) By the binomial theorem with $n = 16$, $a = y$ and $b = -2/y$, each term in the expansion is of the form

$$\begin{aligned} & {}^{16}C_k y^{16-k} \left(-\frac{2}{y}\right)^k \\ & = {}^{16}C_k (-2)^k y^{16-k} \left(\frac{1}{y}\right)^k \\ & = {}^{16}C_k (-2)^k y^{16-k} y^{-k} \\ & = {}^{16}C_k (-2)^k y^{16-2k}. \end{aligned}$$

The term in y^2 is obtained when $16 - 2k = 2$, which gives $k = 7$. Hence the coefficient of y^2 is

$$\begin{aligned} & {}^{16}C_7 (-2)^7 = 11\,440 \times (-2^7) \\ & = -1\,464\,320. \end{aligned}$$

- (e) By the binomial theorem with $n = 5$, $a = x$ and $b = y/x$, each term in the expansion is of the form

$$\begin{aligned} {}^5C_k x^{5-k} \left(\frac{y}{x}\right)^k &= {}^5C_k x^{5-k} (x^{-1})^k y^k \\ &= {}^5C_k x^{5-k} x^{-k} y^k \\ &= {}^5C_k x^{5-2k} y^k. \end{aligned}$$

The term in xy^2 is obtained when $k = 2$ (giving the power of x as $5 - 2k = 1$).
Hence the coefficient of xy^2 is

$${}^5C_2 = 10.$$

Solution to Exercise 33

By the binomial theorem with $n = 10$, and with a replaced by a^3 and b replaced by $-2/a^2$, each term in the expansion is of the form

$$\begin{aligned} {}^{10}C_k (a^3)^{10-k} \left(-\frac{2}{a^2}\right)^k \\ &= {}^{10}C_k (-2)^k a^{30-3k} (a^{-2})^k \\ &= {}^{10}C_k (-2)^k a^{30-3k} a^{-2k} \\ &= {}^{10}C_k (-2)^k a^{30-5k}. \end{aligned}$$

- (a) The term in a^{25} is obtained when $30 - 5k = 25$, which gives $k = 1$. Hence the coefficient of a^{25} is
- $${}^{10}C_1 (-2)^1 = 10 \times (-2) = -20.$$
- (b) The term in a^{-10} is obtained when $30 - 5k = -10$, which gives $k = 8$. Hence the coefficient of a^{-10} is
- $${}^{10}C_8 (-2)^8 = 45 \times 2^8 = 11\,520.$$
- (c) The constant term is the term in which the power of a is zero. This is obtained when $30 - 5k = 0$, which gives $k = 6$.
Hence the constant term is
- $${}^{10}C_6 (-2)^6 = 210 \times 2^6 = 13\,440.$$
- (d) For a term in a^2 , we need $30 - 5k = 2$.
The solution to this equation is not an integer, so there is no term in a^2 .